Newton’s Method as a Formal Recurrence

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Non-commuting Algebra

We have defined $P_n$ and $Q_n$ with $P_0(x) = x$ and $Q_0(x) = 1$. If we instead let $P_0(x) = x$ and $Q_0(x) = y$, it can be verified that we get slightly different formula:

\[ P_n(x, y) = a^{2^{n-1}}x^{2^n} + \sum_{k=0}^{2^n-2} \sum_{i=1}^{2^n-k-1} (-1)^i \binom{2^n}{k} \binom{2^n-k-i-1}{i-1} a^{k+i-1} b^{2^n-k-2i} c^i x^k y^{2^n-k} \]

\[ Q_n(x, y) = \sum_{k=0}^{2^n-1} \sum_{i=0}^{2^n-k-1} (-1)^i \binom{2^n}{k} \binom{2^n-k-i-1}{i} a^{k+i} b^{2^n-k-2i-1} c^i x^k y^{2^n-k} \]

But suppose that $x$ and $y$ do not commute, but rather satisfy the formula $yx = qxy$. What happens? If we define

\[ \left[ \begin{array}{c} n \\ k \end{array} \right]_q = \prod_{i=1}^{n-k} \frac{1 - q^{i+k}}{1 - q^i} \]

then the q-version of the binomial formula is: \footnote{\[ (x + y)^n = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q x^k y^{n-k}. \]}

**Conjecture 1.** If $yx = qxy$ and if $P_n(x, y)$ and $P_n(x, y)$ are defined by

\[
\begin{align*}
P_0 &= x \\
Q_0 &= y \\
P_{n+1} &= aP_n^2 - cQ_n^2 \\
Q_{n+1} &= aP_nQ_n + aQ_nP_n + bQ_nQ_n.
\end{align*}
\]

then

\[ P_n(x, y) = a^{2^{n-1}}x^{2^n} + \sum_{k=0}^{2^n-2} \sum_{i=1}^{2^n-k-1} (-1)^i \left[ \binom{2^n}{k} q \binom{2^n-k-i-1}{i-1} a^{k+i-1} b^{2^n-k-2i} c^i x^k y^{2^n-k} \right] \]

\[ Q_n(x, y) = \sum_{k=0}^{2^n-1} \sum_{i=0}^{2^n-k-1} (-1)^i \left[ \binom{2^n}{k} q \binom{2^n-k-i-1}{i} a^{k+i} b^{2^n-k-2i-1} c^i x^k y^{2^n-k} \right]. \]

It is not clear to us that the proof we provided for Theorem (??) is applicable to this more general conjecture.
References