Newton's Method as a Formal Recurrance

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May 7, 2004

Non-commuting Algebra

We have defined P_n and Q_n with $P_0(x) = x$ and $Q_0(x) = 1$. If we instead let $P_0(x) = x$ and $Q_0(x) = y$, it can be verified that we get slightly different formula:

$$P_n(x,y) = a^{2^n-1}x^{2^n} \;+\; \sum_{k=0}^{(2^n-2)} \sum_{i=1}^{(2^n-k-1)} (-1)^i \binom{2^n}{k}\binom{2^n-k-i-1}{i-1} a^{k+i-1}\, b^{2^n-k-2i}\, c^i\, x^k y^{2^n-k}$$

$$Q_n(x,y) = \sum_{k=0}^{(2^n-1)} \sum_{i=0}^{(2^n-k-1)} (-1)^i \binom{2^n}{k}\binom{2^n-k-i-1}{i} a^{k+i}\, b^{2^n-k-2i-1}\, c^i\, x^k y^{2^n-k}$$

But suppose that x and y do not commute, but rather satisfy the formula $yx = qxy$. What happens? If we define

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{n-k} \frac{1-q^{i+k}}{1-q^i}$$

then the q-version of the binomial formula is: [1]

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}.$$

Conjecture 1. *If $yx = qxy$ and if $P_n(x,y)$ and $P_n(x,y)$ are defined by*

$$P_0 = x \qquad Q_0 = y$$
$$P_{n+1} = aP_n^2 - cQ_n^2 \qquad Q_{n+1} = aP_nQ_n + aQ_nP_n + bQ_nQ_n.$$

then

$$P_n(x,y) = a^{2^n-1}x^{2^n} \;+\; \sum_{k=0}^{(2^n-2)} \sum_{i=1}^{(2^n-k-1)} (-1)^i \begin{bmatrix} 2^n \\ k \end{bmatrix}_q \binom{2^n-k-i-1}{i-1} a^{k+i-1}\, b^{2^n-k-2i}\, c^i\, x^k y^{2^n-k}$$

$$Q_n(x,y) = \sum_{k=0}^{(2^n-1)} \sum_{i=0}^{(2^n-k-1)} (-1)^i \begin{bmatrix} 2^n \\ k \end{bmatrix}_q \binom{2^n-k-i-1}{i} a^{k+i}\, b^{2^n-k-2i-1}\, c^i\, x^k y^{2^n-k}.$$

It is not clear to us that the proof we provided for Theorem (**??**) is aplicable to this more general conjecture.

References

[1] Marcel Paul Schützenberger. Une interprétation de certaines solutions de l'équation fonctionnelle: $F(x+y) = F(x)F(y)$. *C. R. Acad. Sci. Paris*, 236:352–353, 1953.