The Dana Scott Recurrence

Hal Canary

University of Wisconsin–Madison

March 23, 2004

1 Introduction

In an article on the Somos Sequence [1], David Gale mentions a recurrence discovered by Dana Scott,

$$a_n a_{n-4} = a_{n-1} a_{n-3} + a_{n-2} \qquad a_1 = a_2 = a_3 = a_4 = 1, \tag{1}$$

and mentions that there exists a number theoretical proof that it always gives integers. We will present an alternative proof.

Emilie Hogan found a family of similar recurrences indexed by the parameter k (k is odd):

$$a_n a_{n-k} = a_{n-1} a_{n-(k-1)} + a_{n-(k-1)/2} + a_{n-(k+1)/2}. \tag{2}$$

Hopefully, (2) can be approached just like we will approach (1).

2 Linearizing

Equation (1) generates the sequence $\{1,\ 1,\ 1,\ 1,\ 2,\ 3,\ 5, 13,\ 22,\ 41,\ 111,\ 191,\ 361,\ 982, \ldots\}$. Using a computer, we found that this sequence grows like $\mathrm{O}(C^n)$. This suggested that the sequence satisfied a linear recurrence, which we proceeded to find:

$$0 = a_n - 10\ a_{n-3} + 10\ a_{n-6} - a_{n-9}. \tag{3}$$

The characteristic equation of (3) is

$$0 = x^9 - 10\ x^6 + 10\ x^3 - 1.$$

Finding the explicit formula for a linear recurrance is fairly simple, but time-consuming. It is better to ask a computer to do it. The following code will produce the explicit formula.

```
#begin Maple code
# Manually enter the characteristic polynomial.
chari := (x) -> x^9 - 10*x^6 + 10*x^3 - 1:
# order of characteristic polynomial.
N:=9:
# quadratic recurrance.
quadrat := proc(n) option remember;
  if n<3 then 1
  else return((quadrat(n-1)*quadrat(n-3)+
    quadrat(n-2))/quadrat(n-4));
  fi;
end:
```

```
# Solve for roots of characteristic polynomial.
routes := solve(chari(x),x) :
# Get initial conditions from quadratic.
initial := seq(quadrat(n),n=0..N-1):
# Solve for coefficients.
assign(solve({seq( add( coffs[i]*routes[i]^(j-1) ,i=1..N)
  = initial[j],j=1..N)}, {seq(coffs[i],i=1..N)})):
# Explicit function.
explicit := (n) ->
  simplify(add( coffs[i]*routes[i]^n ,i=1..N)):
# Print out explicit formula.
interface(prettyprint=false):
explicit(n);
#verify that explicit satisfies the quadratic recurrence.
evalb(simplify(explicit(n)*explicit(n-4)) =
 simplify(explicit(n-1)*explicit(n-3)+explicit(n-2)));
#end Maple code
```

If the last line of that code returns *true* (which it doesn't), then we have just proved that the Dana Scott Recurrence is equivilant to the recurrence given in (3), and since a linear recurrance that starts with integers always gives integer, the Dana Scott Recurrence wil also always give integers.

3 A Better Proof

Define the sequence $\{a\}$ recursivly:

$$a_n = 10\ a_{n-3} - 10\ a_{n-6} + a_{n-9}. \tag{4}$$

With the intial conditions $(a_1 \ldots a_9) = (1, 1, 1, 1, 2, 3, 5, 13, 22)$.

We wish to prove by induction that $\{a\}$ is the same as the Dana Scott recurrance, that it satisifes

$$a_n a_{n-4} - a_{n-1}a_{n-3} - a_{n-2} = 0.$$

For convienence, let

$$\phi(n) := a_n a_{n-4} - a_{n-1}a_{n-3} - a_{n-2}$$

Assume that $\phi(k) = 0$ for $k < n$. Show that $\phi(n) = 0$. This gives:

$$\begin{aligned}
\phi(n-1) &= a_{n-1}a_{n-5} - a_{n-2}a_{n-4} - a_{n-3} = 0 \\
\phi(n-2) &= a_{n-2}a_{n-6} - a_{n-3}a_{n-5} - a_{n-4} = 0 \\
\phi(n-3) &= a_{n-3}a_{n-7} - a_{n-4}a_{n-6} - a_{n-5} = 0 \\
\phi(n-4) &= a_{n-4}a_{n-8} - a_{n-5}a_{n-7} - a_{n-6} = 0 \\
\phi(n-5) &= a_{n-5}a_{n-9} - a_{n-6}a_{n-8} - a_{n-7} = 0 \\
\phi(n-6) &= a_{n-6}a_{n-10} - a_{n-7}a_{n-9} - a_{n-8} = 0 \\
\phi(n-7) &= a_{n-7}a_{n-11} - a_{n-8}a_{n-10} - a_{n-9} = 0 \\
\phi(n-8) &= a_{n-8}a_{n-12} - a_{n-9}a_{n-11} - a_{n-10} = 0 \\
\phi(n-9) &= a_{n-9}a_{n-13} - a_{n-10}a_{n-12} - a_{n-11} = 0 \\
\phi(n-10) &= a_{n-10}a_{n-14} - a_{n-11}a_{n-13} - a_{n-12} = 0 \\
&\cdots
\end{aligned}$$

Now, compute $\phi(n)$.

$$\phi(n) = a_n a_{n-4} - a_{n-1} a_{n-3} - a_{n-2}$$

Substitute for a_n and a_{n-1} from the definition of $\{a\}$.

$$\begin{aligned} a_n &= 10a_{n-3} - 10a_{n-6} + a_{n-9} \\ a_{n-1} &= 10a_{n-4} - 10a_{n-7} + a_{n-10} \end{aligned}$$

$$\begin{aligned} \phi(n) &= (10a_{n-3} - 10a_{n-6} + a_{n-9})a_{n-4} \\ &\quad -(10a_{n-4} - 10a_{n-7} + a_{n-10})a_{n-3} \\ &\quad -(10a_{n-5} - 10a_{n-8} + a_{n-11}) \end{aligned}$$

Simplify:

$$\begin{aligned} \phi(n) &= 10a_{n-3}a_{n-7} - 10a_{n-4}a_{n-6} - 10a_{n-5} \\ &\quad +a_{n-4}a_{n-9} - a_{n-3}a_{n-10} + 10a_{n-8} - a_{n-11} \end{aligned}$$

Since $\phi(n-3) = 0$,

$$\begin{aligned} \phi(n) &= a_{n-4}a_{n-9} - a_{n-3}a_{n-10} \\ &\quad +10a_{n-8} - a_{n-11} \end{aligned}$$

Substitute for a_{n-3} and a_{n-4} from the definition of $\{a\}$.

$$\begin{aligned} a_{n-3} &= 10a_{n-6} - 10a_{n-9} + a_{n-12} \\ a_{n-4} &= 10a_{n-7} - 10a_{n-10} + a_{n-13} \end{aligned}$$

$$\begin{aligned} \phi(n) &= (10a_{n-7} - 10a_{n-10} + a_{n-13})a_{n-9} \\ &\quad -(10a_{n-6} - 10a_{n-9} + a_{n-12})a_{n-10} \\ &\quad +10a_{n-8} \\ &\quad -a_{n-11}; \end{aligned}$$

Simplify:

$$\begin{aligned} \phi(n) &= -10a_{n-10}a_{n-6} + 10a_{n-9}a_{n-7} + 10a_{n-8} \\ &\quad +a_{n-9}a_{n-13} - a_{n-10}a_{n-12} - a_{n-11} \end{aligned}$$

$$\begin{aligned} \phi(n) &= -10(+a_{n-10}a_{n-6} - a_{n-9}a_{n-7} - a_{n-8}) \\ &\quad +a_{n-9}a_{n-13} - a_{n-10}a_{n-12} - a_{n-11} \end{aligned}$$

Since $\phi(n-6) = \phi(n-9) = 0$

$$a_{n-6}a_{n-10} - a_{n-7}a_{n-9} - a_{n-8} = 0$$

$$a_{n-9}a_{n-13} - a_{n-10}a_{n-12} - a_{n-11} = 0$$

$$\phi(n) = 0$$

QED.

4 Aside: Laurent Polynomials

If we choose the first four terms of (1) to be (w, x, y, z) instead of $(1, 1, 1, 1)$, then the next five terms are

$$a_5 = \frac{zx + y}{w}$$

$$a_6 = \frac{yzx + y^2 + zw}{wx}$$

$$a_7 = \frac{yz^2x + zy^2 + z^2w + zx^2 + xy}{wxy}$$

$$a_8 = \frac{yz^3x^2 + 2y^2z^2x + zy^3 + z^3wx + z^2wy + z^2x^3 + 2zx^2y + xy^2 + wy^2zx + wy^3 + w^2yz}{w^2xyz}$$

$$a_9 = \frac{\begin{array}{c} yz^2x^3 + x^2y^2z^3 + x^2z^2w + 2x^2zy^2 + x^2zw^2 + 2xwyz^3 + 2xz^2y^3 \\ +xz^2w^2y + xzwy + xzwy^3 + xy^3 + w^2z^3 + 2z^2y^2w + z^2w^3 + 2zw^2y^2 + zy^4 + wy^4 \end{array}}{x^2yzw^2}$$

These are all Laurent polynomials, so if we choose the first nine terms of $\{a\}$ to be $(w, x, y, z, a_5, a_6, a_7, a_8, a_9)$ instead of $(1, 1, 1, 1, 2, 3, 5, 13, 22)$, then by the linearity of (4), all subsequent terms are Laurent polynomials.

```
#begin Maple code
a := proc(n)
  if n = 1 then w
  elif n = 2 then x
  elif n = 3 then y
  elif n = 4 then z
  else simplify((a(n - 1)*a(n - 3) + a(n - 2))/a(n - 4))
  end if
end proc:
interface(prettyprint=false);
seq(a(n),n=1..9);
#end Maple code
```

References

[1] David Gale. "The Strange and Surprising Saga of the Somos Sequences." *Mathematical Intelligencer* **13** 1 (1991) 40-42.