1 Introduction

In an article on the Somos Sequence [1], David Gale mentions a recurrence discovered by Dana Scott,

\[ a_n a_{n-4} = a_{n-1} a_{n-3} + a_{n-2} \quad a_1 = a_2 = a_3 = a_4 = 1, \quad (1) \]

and mentions that there exists a number theoretical proof that it always gives integers. We will present an alternative proof.

Emilie Hogan found a family of similar recurrences indexed by the parameter \( k \) (\( k \) is odd):

\[ a_n a_{n-k} = a_{n-1} a_{n-(k-1)} + a_{n-(k-1)/2} + a_{n-(k+1)/2}. \quad (2) \]

Hopefully, (2) can be approached just like we will approach (1).

2 Linearizing

Equation (1) generates the sequence \{1, 1, 1, 1, 2, 3, 5, 13, 22, 41, 111, 191, 361, 982, \ldots\}. Using a computer, we found that this sequence grows like \( O(C^n) \). This suggested that the sequence satisfied a linear recurrence, which we proceeded to find:

\[ 0 = a_n - 10 a_{n-3} + 10 a_{n-6} - a_{n-9}. \quad (3) \]

The characteristic equation of (3) is

\[ 0 = x^9 - 10 x^6 + 10 x^3 - 1. \]

Finding the explicit formula for a linear recurrence is fairly simple, but time-consuming. It is better to ask a computer to do it. The following code will produce the explicit formula.

```maple
# begin Maple code
# Manually enter the characteristic polynomial.
chari := (x) -> x^9 - 10*x^6 + 10*x^3 - 1:
# order of characteristic polynomial.
N:=9:
# quadratic recurrance.
quadrat := proc(n) option remember;
if n<3 then 1
else return((quadrat(n-1)*quadrat(n-3)+quadrat(n-2))/quadrat(n-4));
fi;
end:
```

1
# Solve for roots of characteristic polynomial.
routes := solve(chari(x),x):

# Get initial conditions from quadratic.
initial := seq(quadrat(n),n=0..N-1):

# Solve for coefficients.
assign(solve({seq( add( coffs[i]*routes[i]^(j-1) ,i=1..N)
    = initial[j],j=1..N}},{seq(coffs[i],i=1..N)})):

# Explicit function.
explicit := (n) ->
simplify(add( coffs[i]*routes[i]^n ,i=1..N)):

# Print out explicit formula.
interface(prettyprint=false):
explicit(n);

#verify that explicit satisfies the quadratic recurrence.
evalb(simplify(explicit(n)*explicit(n-4))
    = simplify(explicit(n-1)*explicit(n-3)+explicit(n-2)));

#end Maple code

If the last line of that code returns true (which it doesn’t), then we have just proved that the Dana Scott Recurrence is equivalent to the recurrence given in [3], and since a linear recurrence that starts with integers always gives integer, the Dana Scott Recurrence will also always give integers.

3 A Better Proof

Define the sequence \( \{a\} \) recursively:

\[
a_n = 10a_{n-3} - 10a_{n-6} + a_{n-9}.
\]

With the initial conditions \((a_1 \ldots a_9) = (1, 1, 1, 1, 2, 3, 5, 13, 22)\).

We wish to prove by induction that \( \{a\} \) is the same as the Dana Scott recurrence, that it satisfies

\[
a_n a_{n-4} - a_{n-1} a_{n-3} - a_{n-2} = 0.
\]

For convenience, let

\[
\phi(n) := a_n a_{n-4} - a_{n-1} a_{n-3} - a_{n-2}
\]

Assume that \( \phi(k) = 0 \) for \( k < n \). Show that \( \phi(n) = 0 \). This gives:

\[
\begin{align*}
\phi(n - 1) &= a_{n-1}a_{n-5} - a_{n-2}a_{n-4} - a_{n-3} = 0 \\
\phi(n - 2) &= a_{n-2}a_{n-6} - a_{n-3}a_{n-5} - a_{n-4} = 0 \\
\phi(n - 3) &= a_{n-3}a_{n-7} - a_{n-4}a_{n-6} - a_{n-5} = 0 \\
\phi(n - 4) &= a_{n-4}a_{n-8} - a_{n-5}a_{n-7} - a_{n-6} = 0 \\
\phi(n - 5) &= a_{n-5}a_{n-9} - a_{n-6}a_{n-8} - a_{n-7} = 0 \\
\phi(n - 6) &= a_{n-6}a_{n-10} - a_{n-7}a_{n-9} - a_{n-8} = 0 \\
\phi(n - 7) &= a_{n-7}a_{n-11} - a_{n-8}a_{n-10} - a_{n-9} = 0 \\
\phi(n - 8) &= a_{n-8}a_{n-12} - a_{n-9}a_{n-11} - a_{n-10} = 0 \\
\phi(n - 9) &= a_{n-9}a_{n-13} - a_{n-10}a_{n-12} - a_{n-11} = 0 \\
\phi(n - 10) &= a_{n-10}a_{n-14} - a_{n-11}a_{n-13} - a_{n-12} = 0 \\
\end{align*}
\]

...
Now, compute $\phi(n)$.

$$\phi(n) = a_na_{n-4} - a_{n-1}a_{n-3} - a_{n-2}$$

Substitute for $a_n$ and $a_{n-1}$ from the definition of $\{a\}$.

$$a_n = 10a_{n-3} - 10a_{n-6} + a_{n-9}$$
$$a_{n-1} = 10a_{n-4} - 10a_{n-7} + a_{n-10}$$

$$\phi(n) = (10a_{n-3} - 10a_{n-6} + a_{n-9})a_{n-4}$$
$$-10a_{n-4} - 10a_{n-7} + a_{n-10}a_{n-3}$$
$$-10a_{n-5} - 10a_{n-8} + a_{n-11}$$

Simplify:

$$\phi(n) = 10a_{n-3}a_{n-7} - 10a_{n-4}a_{n-6} - 10a_{n-5}$$
$$+a_{n-4}a_{n-9} - a_{n-3}a_{n-10} + 10a_{n-8} - a_{n-11}$$

Since $\phi(n - 3) = 0$,

$$\phi(n) = a_{n-4}a_{n-9} - a_{n-3}a_{n-10}$$
$$+10a_{n-8} - a_{n-11}$$

Substitute for $a_{n-3}$ and $a_{n-4}$ from the definition of $\{a\}$.

$$a_{n-3} = 10a_{n-6} - 10a_{n-9} + a_{n-12}$$
$$a_{n-4} = 10a_{n-7} - 10a_{n-10} + a_{n-13}$$

$$\phi(n) = (10a_{n-7} - 10a_{n-10} + a_{n-13})a_{n-9}$$
$$-10a_{n-6} - 10a_{n-9} + a_{n-12}a_{n-10}$$
$$+10a_{n-8}$$
$$-a_{n-11};$$

Simplify:

$$\phi(n) = -10a_{n-10}a_{n-6} + 10a_{n-9}a_{n-7} + 10a_{n-8}$$
$$+a_{n-9}a_{n-13} - a_{n-10}a_{n-12} - a_{n-11}$$

$$\phi(n) = -10( +a_{n-10}a_{n-6} - a_{n-9}a_{n-7} - a_{n-8})$$
$$+a_{n-9}a_{n-13} - a_{n-10}a_{n-12} - a_{n-11}$$

Since $\phi(n - 6) = \phi(n - 9) = 0$

$$a_{n-9}a_{n-10} - a_{n-7}a_{n-9} - a_{n-8} = 0$$
$$a_{n-9}a_{n-13} - a_{n-10}a_{n-12} - a_{n-11} = 0$$

$$\phi(n) = 0$$

QED.
4 Aside: Laurent Polynomials

If we choose the first four terms of \([1]\) to be \((w, x, y, z)\) instead of \((1, 1, 1, 1)\), then the next five terms are

\[
\begin{align*}
a_5 &= \frac{zx + y}{w} \\
a_6 &= \frac{yzx + y^2 + zw}{wx} \\
a_7 &= \frac{yz^2x + zy^2 + z^2w + zx^2 + xy}{wxy} \\
a_8 &= \frac{yz^3x^2 + 2y^2z^2x + zy^3 + z^3wx + z^2wy + z^2x^3 + 2zxy^2 + xy^2 + wy^2zx + wy^3 + w^2yz}{w^2xyz} \\
a_9 &= \frac{yz^2x^3 + x^2y^3 + x^2z^2w + 2x^2zy^2 + x^2zw^2 + 2xwyz^3 + 2xz^2y^3 + xz^2w^3 + 2z^2y^2w^2 + 2z^2w^2y^2 + zy^4 + wy^4}{x^2yzw^2}
\end{align*}
\]

These are all Laurent polynomials, so if we choose the first nine terms of \(\{a\}\) to be \((w, x, y, z, a_5, a_6, a_7, a_8, a_9)\) instead of \((1, 1, 1, 2, 3, 5, 13, 22)\), then by the linearity of \([4]\), all subsequent terms are Laurent polynomials.

\[
\text{#begin Maple code}
\text{a := proc(n)}
\text{if n = 1 then w}
\text{elif n = 2 then x}
\text{elif n = 3 then y}
\text{elif n = 4 then z}
\text{else simplify((a(n-1)*a(n-3) + a(n-2))/a(n-4))}
\text{end if}
\text{end proc;}
\text{interface(prettyprint=false);}
\text{seq(a(n),n=1..9);}
\text{#end Maple code}
\]

References