Newton's Recurrence

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Abstract

If Newton's method is repeatedly applied, then something cool happens.

1 Introduction

Newton's method is an algorithm that allows one to approximate the roots of a function, starting with a guess.

If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function, then Newton's method maps one approximation to a better approximation.

$$N : x \mapsto x - \frac{f(x)}{f'(x)}.$$

$$\alpha a$$

Definition a sequence $\{x\}$ using N:

$$x_n = N(x_{n-1}) = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}.$$

Instead of evaluating $N(x)$ at each iteration of the method, leave it in terms of the original guess, x_0. Let f be the quadratic polynomial $f(x) = a(x-r_1)(x-r_2)$. Then the first two functions in the sequence are:

$$x_1 = N(x_0) = \frac{-x_0^2 + r_1 r_2}{-2x_0 + r_2 + r_1}$$

$$x_2 = N^2(x_0) = \frac{-x_0^4 + 6r_1r_2x_0^2 - 4r_1r_2^2x_0 + r_1r_2^3 - 4r_1^2r_2x_0 + r_1^3r_2 + r_1^2r_2^2}{(2x_0^2 - 2x_0r_2 + r_2^2 - 2x_0r_1 + r_1^2)(-2x_0 + r_2 + r_1)}$$

Theorem 1 *The nth iteration of Newton's Recurrence is*

$$x_n = N^n(x_0) = \frac{\sum_{k=1}^{2^n} (-1)^{2^n} \binom{2^n}{k} \left(r_1 r_2^{2^n-k} - r_2 r_1^{2^n-k} \right) x_0^k}{\sum_{k=1}^{2^n} (-1)^{2^n} \binom{2^n}{k} \left(r_2^{2^n-k} - r_1^{2^n-k} \right) x_0^k}.$$

Theorem 2 *If $f(x) = ax^2 + bx + c$, then the nth iteration of Newton's Recurrence is*

$$x_n = N^n(x_0) = \frac{\sum \left(?? \right) x_0^k}{\sum \left(?? \right) x_0^k}.$$

2 Möbius Transformation

Here, we will prove Theorem 1. Define a Möbius Transformation M as

$$M : z \mapsto \frac{z - r_1}{z - r_2}$$

We claim that $N(x) = M^{-1}(M(x)^2)$, or that the diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{N_f} & \mathbb{R} \\ {\scriptstyle M}\downarrow & & \downarrow{\scriptstyle M} \\ \widetilde{\mathbb{C}} & \xrightarrow[(\cdot)^2]{} & \widetilde{\mathbb{C}} \end{array}$$

commutes.