

# Newton's Recurrence

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## Abstract

If Newton's method is repeatedly applied, then something cool happens.

## 1 Introduction

Newton's method is an algorithm that allows one to approximate the roots of a function, starting with a guess.

If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a differentiable function, then Newton's method maps one approximation to a better approximation.

$$N : x \mapsto x - \frac{f(x)}{f'(x)}.$$

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Definition a sequence  $\{x\}$  using  $N$ :

$$x_n = N(x_{n-1}) = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}.$$

Instead of evaluating  $N(x)$  at each iteration of the method, leave it in terms of the original guess,  $x_0$ . Let  $f$  be the quadratic polynomial  $f(x) = a(x-r_1)(x-r_2)$ . Then the first two functions in the sequence are:

$$x_1 = N(x_0) = \frac{-x_0^2 + r_1 r_2}{-2x_0 + r_2 + r_1}$$

$$x_2 = N^2(x_0) = \frac{-x_0^4 + 6r_1 r_2 x_0^2 - 4r_1 r_2^2 x_0 + r_1 r_2^3 - 4r_1^2 r_2 x_0 + r_1^3 r_2 + r_1^2 r_2^2}{(2x_0^2 - 2x_0 r_2 + r_2^2 - 2x_0 r_1 + r_1^2)(-2x_0 + r_2 + r_1)}$$

**Theorem 1** *The  $n$ th iteration of Newton's Recurrence is*

$$x_n = N^n(x_0) = \frac{\sum_{k=1}^{2^n} (-1)^{2^n} \binom{2^n}{k} (r_1 r_2^{2^n-k} - r_2 r_1^{2^n-k}) x_0^k}{\sum_{k=1}^{2^n} (-1)^{2^n} \binom{2^n}{k} (r_2^{2^n-k} - r_1^{2^n-k}) x_0^k}.$$

**Theorem 2** *If  $f(x) = ax^2 + bx + c$ , then the  $n$ th iteration of Newton's Recurrence is*

$$x_n = N^n(x_0) = \frac{\sum (??) x_0^k}{\sum (??) x_0^k}.$$

## 2 Möbius Transformation

Here, we will prove Theorem 1. Define a Möbius Transformation  $M$  as

$$M : z \mapsto \frac{z - r_1}{z - r_2}$$

We claim that  $N(x) = M^{-1}(M(x)^2)$ , or that the diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{N_f} & \mathbb{R} \\ M \downarrow & & \downarrow M \\ \tilde{\mathbb{C}} & \xrightarrow{(\cdot)^2} & \tilde{\mathbb{C}} \end{array}$$

commutes.